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The non-Abelian Debye screening length beyond leading order

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Abstract

In quantum electrodynamics, static electric fields are screened at non-zero temperatures by charges in the plasma. The inverse screening length, or Debye mass, may be analyzed in perturbation theory and is of order eT at relativistic temperatures. An analogous situation occurs when non-Abelian gauge theories are studied perturbatively, but the perturbative analysis breaks down when corrections of order e^2T are considered. At this order, the Debye mass depends on the non-perturbative physics of confinement, and a perturbative "definition" of the Debye mass as the pole of a gluon propagator does not even make sense. In this work, we show how the Debye mass can be defined non-perturbatively in a manifestly gauge invariant manner (in vector-like gauge theories with zero chemical potential). In addition, we show how the $O(e^2T)$ correction could be determined by a fairly simple, three-dimensional, numerical lattice calculation of the perimeter-law behavior of large, adjoint-charge Wilson loops.

I. INTRODUCTION

Electrically charged particles in a hot plasma react to electromagnetic fields and cause screening of static electric fields at large distances. The inverse screening length, also known as the Debye mass $m_{\rm d}$, may be computed in QED by considering the exchange of a single virtual photon between two static test charges, as depicted in Fig. 1.

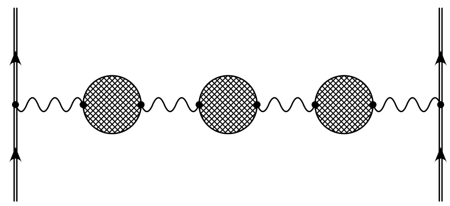

FIG. 1. A single virtual photon exchanged between two static test charges.

The long distance fall-off of the static potential is determined by the position of the pole in the photon propagator at zero frequency. This is given by the solution $\underset{\sim}{p}^2 = -m_{\rm d}^2$ to

$$\underset{\sim}{p}^2 + \Pi_{00}(0, \underset{\sim}{p}) = 0 \,, \tag{1.1}$$

where $\Pi_{\mu\nu}(p_0, \underset{\sim}{p})$ is the self-energy of the photon. In the ultrarelativistic limit (when particle masses and chemical potentials are negligible), the leading-order result is easily computed from the one-loop graph of Fig. 2 and yields

$$m_{\rm d} = \frac{eT}{\sqrt{3}} + O(e^2 T) \tag{1.2}$$

for a theory with a single fermion of charge e.[1]

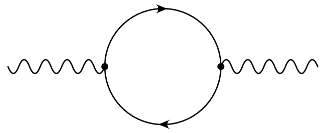

FIG. 2. One-loop self-energy of a photon.

[1] For reviews, see refs. [1,2].

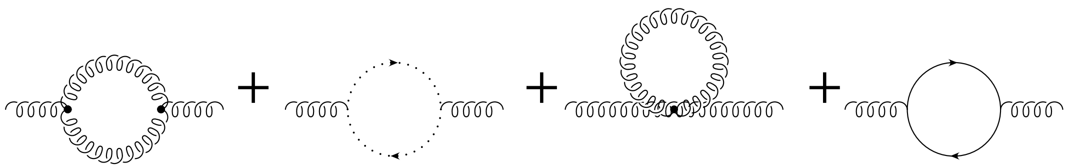

FIG. 3. One-loop self-energy of a gluon.

Unlike electric fields, magnetic fields are unscreened, which is reflected by the fact that

$$\lim_{\underset{\sim}{p}\to 0} \Pi_{ij}(0,\underset{\sim}{p}) = 0\,, \qquad i,j = 1,2,3. \tag{1.3}$$

A similar perturbative calculation carried out in non-Abelian gauge theories, using the one-loop graphs of Fig. 3, yields a lowest-order result of

$$m_{\rm d} = m_0 + O(g^2 T)\,, \tag{1.4a}$$

where

$$m_0 = \left(\frac{N}{3} + \frac{n_{\rm f}}{6}\right)^{1/2} gT \tag{1.4b}$$

for SU(N) gauge theory with $n_{\rm f}$ Dirac fermions.

For the sake of better understanding the nature and reliability of perturbation theory at finite temperature, there has long been an interest in computing the leading correction to this result [3]. It is known, however, that this correction cannot be computed perturbatively in non-Abelian gauge theories [4,5]. As we shall review below, the $O(g^2T)$ correction to the Debye mass receives contributions from fundamentally non-perturbative physics associated with the interactions, at high temperature, of magnetic gluons with momenta of order g^2T. The best that can be done perturbatively is the extraction of a logarithm at that order [5,7]:

$$m_{\rm d} = m_0 + \frac{1}{4\pi} N g^2 T \ln\left(\frac{m_0}{g^2 T}\right) + c\, g^2 T + O(g^3 T)\,. \tag{1.5}$$

The constant c, however, is not computable by perturbation theory.

Because the physics of the $O(g^2T)$ correction is non-perturbative, it behooves us to formulate a non-perturbative definition of what we mean by the Debye mass in the first place. Such a definition should be gauge invariant and preferably implementable in numerical

lattice simulations. The definition (1.1) is unfortunate in both these respects. In particular, the self-energy $\Pi_{\mu\nu}$ is not itself gauge invariant in non-Abelian theories. There are formal proofs that the pole position is gauge invariant order by order in perturbation theory [8], but this is of limited use since perturbation theory breaks down beyond leading order. We should look instead for a definition that is *manifestly* gauge invariant.

The purpose of this paper is two-fold: (*i*) to give a natural non-perturbative definition of the Debye mass, and (*ii*) to show how the constant c in the expansion (1.5) could be extracted from a relatively simple numerical computation of the perimeter law fall-off of large, adjoint-charge Wilson loops in three-dimensional, zero-temperature, pure lattice gauge theory. In section II, we briefly review the source of the breakdown of perturbation theory. In section III, we construct a manifestly gauge invariant, non-perturbative definition of the Debye mass. We review why one method sometimes suggested in the literature—extracting the Debye mass from the long-distance correlation of Wilson lines—is inadequate. Our definition works only for vector-coupled gauge theories, such as QCD or QED, and only at zero chemical potential. We explain what the difficulties are for axially-coupled theories or non-zero chemical potentials, and we outline the problems with making a non-perturbative definition of the Debye mass in those cases. Finally, section IV contains our derivation of the $O(g^2T)$ correction to the Debye mass in terms of three-dimensional Wilson loops.

II. NON-PERTURBATIVE HIGH TEMPERATURE PHYSICS

The physical picture behind our definition will be clearer if we first review the source of non-perturbative effects in hot non-Abelian gauge theories.[2] The problem is easiest to understand by considering a series of effective theories corresponding to larger and larger distance scales in the hot plasma. Since we are interested in studying the screening of *static* electric fields, we can work directly in Euclidean space where non-zero temperature

[2]More details can be found in the discussions and reviews of refs. [1,9,10,11].

corresponds to making the Euclidean time direction periodic with period $\beta = 1/T$. So

$$Z = \int [\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}A] \exp\left(-\frac{1}{g^2}\int_0^\beta d\tau\, d^3x\, \mathcal{L}_{\rm E}\right) , \tag{2.1}$$

where we have suppressed details of ghosts and gauge-fixing. Boson (fermion) fields have (anti-)periodic boundary conditions in Euclidean time. At distances large compared to β, the dynamics of the time direction decouples, and one obtains an effective *three*-dimensional theory of the zero-frequency modes of the original fields. Since the fermionic fields are anti-periodic, only the bosonic degrees of freedom are relevant in this effective theory.[3] Schematically,

$$Z \to \int [\mathcal{D}A] \exp\left(-\frac{1}{g^2 T}\int d^3x\, \mathcal{L}_{\rm eff}\right) . \tag{2.2}$$

The effective theory (2.2) is a three-dimensional gauge field $\underset{\sim}{A}$ coupled to a three-dimensional adjoint-charge scalar corresponding to A_0.[4] One of the effects of integrating out physics with momenta of order T is that the adjoint scalar obtains a mass of order gT. As indicated in (2.2), the gauge coupling constant in the three-dimensional theory is $g_3^2 \equiv g^2 T$.

Next, consider distances in the three-dimensional theory that are large compared to $1/gT$. At these distances the adjoint scalar decouples, and the new effective theory is a pure gauge theory in three dimensions.[5] Three dimensional non-Abelian gauge theories are confining. Moreover, the only remaining parameter of the theory is the three-dimensional coupling g^2T and so, by dimensional analysis, the confinement radius is of order $1/g^2T$. The

[3]Physically, light bosons dominate over light fermions at low frequency because of the infrared divergence of the Bose distribution $1/(e^{\beta E}-1)$ as $E\to 0$.

[4]Purists may object to saying that the adjoint scalar in the effective three-dimensional theory arises from A_0 in the four-dimensional theory, because this statement is gauge dependent. The gauge independent identification is that the three-dimensional scalar corresponds to the traceless part of the path ordered exponential $\mathcal{P}\exp i\int_0^\beta A_0(\tau,\underset{\sim}{x})\,d\tau$ in the original four-dimensional theory. However, we shall continue to refer to the adjoint scalar simply as A_0.

[5]There can be scalars too if they are part of the fundamental theory, such as for electroweak theory, and if the temperature is fine-tuned to be near a phase transition.

physics of magnetic gluons with momenta of order g^2T is therefore non-perturbative, and the physical states of the three-dimensional effective theory are glue-balls rather than individual gluons. This is unlike the case in zero-temperature *four*-dimensional theories, where the confinement radius diverges exponentially as $g \to 0$ and non-perturbative contributions are never the same order as perturbative ones.

It is important to keep in mind that the physics at large distances is the physics of three-dimensional *confinement*, and that it is this confinement which cuts off infrared divergences encountered in perturbation theory. Some papers in the literature work under the misapprehension that the infrared physics is instead cut off by some sort of mass of order g^2T for the gauge field $\utilde{A}$. This is as misleading as thinking of confinement in zero-temperature QCD as being described by a gluon mass. A mass would cause large (fundamental-charge) spatial Wilson loops in high-temperature gauge theory to have perimeter-law behavior because it would screen the gauge force. Instead, such loops will have area-law behavior—the signal of confinement.

III. DEFINING THE DEBYE MASS

Begin by considering QED. The Debye mass can simply be defined by the correlation length of the equal-time electric field correlation function:

$$\langle \utilde{E}(\utilde{x}) \cdot \utilde{E}(0) \rangle \sim e^{-m_{\rm d}|\utilde{x}|}/|\utilde{x}|^3 \qquad \text{as } |\utilde{x}| \to \infty, \tag{3.1a}$$

or

$$m_{\rm d} \equiv - \lim_{|\utilde{x}| \to \infty} |\utilde{x}|^{-1} \ln \langle \utilde{E}(\utilde{x}) \cdot \utilde{E}(0) \rangle \,. \tag{3.1b}$$

This is equivalent to the definition (1.1) in terms of the photon self-energy because the exponential rate of decay of a propagator at large distance is determined by the location of

singularities nearest the real axis in momentum space.[6]

Unfortunately, this is a poor definition in non-Abelian theories because $\underset{\sim}{E}$ is no longer gauge invariant. One might instead consider a definition in terms of the correlation between two static test charges. Specifically, a manifestly gauge-invariant possibility would be to define the Debye screening length as the correlation length between Wilson lines (also known as Polyakov loops),

$$\langle L(\underset{\sim}{x}) L^{\dagger}(0) \rangle \overset{?}{\sim} e^{-m_{\rm d}|\underset{\sim}{x}|}/|\underset{\sim}{x}| \qquad \text{as } |\underset{\sim}{x}| \to \infty \,, \tag{3.2}$$

where the Wilson line

$$L(\underset{\sim}{x}) \equiv \operatorname{tr} \mathcal{P} \exp i \int_0^{\beta} A_0(\tau, \underset{\sim}{x}) \, d\tau \tag{3.3}$$

is the trace (in the fundamental representation) of the path-ordered exponential of the line integral of the gauge field around the periodic Euclidean space. ($\mathcal{P}$ denotes path ordering.)

Although this definition has occasionally been suggested in the literature, it is wrong. Even in QED, it fails to isolate the quantity one wants to identify as the Debye mass. Though Wilson lines couple directly only to electric fields, they couple *indirectly* to magnetic fields through interactions, and magnetic fields are not screened. Fig. 4 shows how two Wilson lines can exchange a pair of magnetic photons in QED, and so, despite the screening of electric fields, the correlation (3.2) falls off algebraically instead of exponentially.[7] In non-Abelian gauge theory, the coupling to the spatial gauge field can be even more direct [4,7], as in Fig. 5. The non-Abelian case is slightly different from QED, however, because three-dimensional confinement implies that the Wilson lines cannot exchange a *massless* pair of

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[6] The fact that (3.1) specifies coincident times while (1.1) refers to zero frequency makes no difference. All contributions from the (discrete) non-zero frequencies to the equal time correlation function decay as $O(e^{-2\pi T|\underset{\sim}{x}|})$ or faster; hence the zero frequency component dominates at large distance.

[7] The potential between arbitrarily heavy test charges, $V(\underset{\sim}{x}) = -\beta^{-1} \ln \langle L(\underset{\sim}{x}) L^{\dagger}(0) \rangle$, decreases as $|\underset{\sim}{x}|^{-6}$, reflecting a magnetic Van der Waals interaction between the two electron-positron clouds screening the test charges. An analogous case of algebraic screening in non-relativistic theories at finite density is discussed in ref. [6].

magnetic gluons; the pair will instead form a glueball with a mass of order g^2T, and so the correlation length of Wilson lines defined by (3.2) will be of order $1/g^2T$ [4,7]. Regardless, this is not the physics of electric screening.

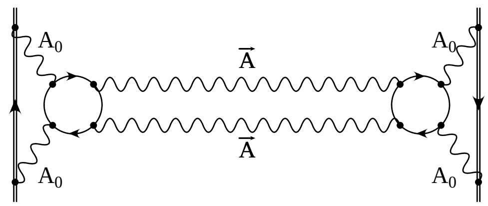

FIG. 4. Power-law interaction between two Wilson lines, representing static test charges, due to the exchange of a pair of unscreened, magnetic photons.

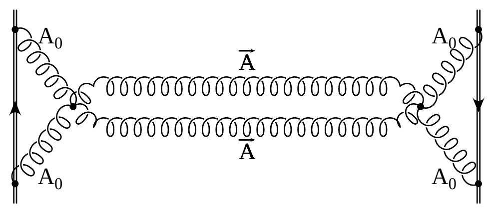

FIG. 5. An exchange of magnetic gluons between two Wilson lines.

Fortunately, there is a simple symmetry which can be used to exclude the unwanted exchange of a pair of magnetic photons or a magnetic glueball: Euclidean time reflection. Euclidean time reflection corresponds to what in real-time is called $\mathcal{TC}$, or time reversal times charge conjugation,[8] and the crucial property is that A_0 is intrinsically odd under this symmetry while the spatial gauge field $\underset{\sim}{A}$ is even. The Euclidean description is more convenient for our purposes, and we shall frequently refer to the symmetry simply as "time reflection." (The reader should note that in Euclidean functional integrals time reflection is no more subtle a symmetry than spatial reflection; there is no extra complication associated with anti-unitarity.) Euclidean time reflection is a useful symmetry because, in the effective three-dimensional theory, the *only* effect it has is to negate the adjoint scalar A_0.

[8]A mnemonic for this fact: In real time, $\mathcal{CPT}$ must be a symmetry of any Lorentz invariant (and unitary) theory. In Euclidean space, $\mathcal{PR}_\tau$, where $\mathcal{R}_\tau$ denotes Euclidean time reflection, is a pure rotation and must be a symmetry of any Euclidean invariant theory. So $\mathcal{R}_\tau$ must correspond to $\mathcal{TC}$, since $\mathcal{P}$ is time independent.

If one considers the correlation of a pair of time-reflection odd operators, instead of the Wilson lines, then the zero-frequency magnetic contributions of the type depicted in figs. 4 and 5 will be eliminated. The lightest intermediate states which can contribute will be those containing a single A_0 (plus surrounding glue), so that the correlation length will be $m_0 + O(g^2 T)$. Any local, gauge-invariant, time-reflection odd operator can be considered as a replacement for the Wilson line, leading to a general definition of the Debye mass:

> *Definition.* Consider the correlation lengths defined by the fall-off, at large spatial separation, of the correlation $\langle A(\underset{\sim}{x}) B(0) \rangle$ between operators A and B that are local (in 3-space), gauge invariant, and odd under Euclidean time reflection (*i.e.*, real-time $\mathcal{TC}$). The inverse Debye screening mass $1/m_{\rm d}$ is the largest such correlation length.

We are thus able to define the Debye mass directly in terms of the long-distance fall-off of certain correlation functions. This definition will only work, however, in theories where real-time $\mathcal{TC}$ is a good symmetry; otherwise, there is nothing to prevent states with a single A_0 from mixing with $\underset{\sim}{A}$ glueballs, and all of our inverse correlation lengths will again be $O(g^2 T)$ instead of $O(gT)$ and will be unrelated to the physics of electric screening. The restriction to $\mathcal{TC}$-conserving theories means that the Debye mass cannot be rigorously defined by the long-distance fall-off of correlation functions in theories with axial couplings, such as electroweak theory, or in the presence of a non-zero chemical potential. We shall comment again on these cases later, but for now our discussion will be restricted to vector-coupled theories, such as QCD, at zero chemical potential.

Before proceeding further, it will be convenient to rephrase our definition in alternative language. Suppose that the separation $\underset{\sim}{x}$ of our operators is in the z direction. In Euclidean space, there is nothing that distinguishes the time dimension as fundamentally different from the spatial ones. One may turn one's head on the side and interchange the labels z and t, as depicted in Fig. 6. One then interprets the original four-dimensional field theory as a zero-temperature theory with one periodic spatial dimension, instead of a finite-temperature

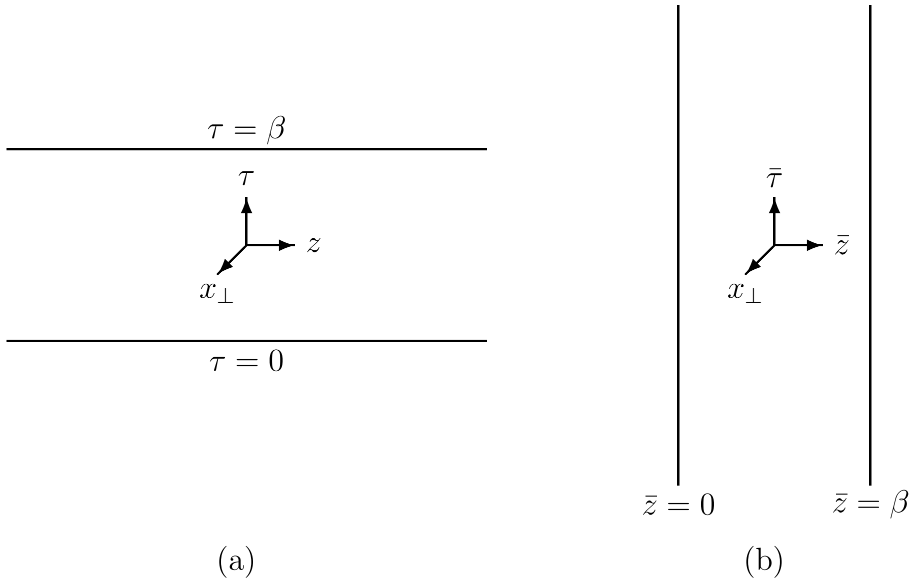

FIG. 6. Interpreting Euclidean time as a periodic spatial direction, by relabeling coordinates (τ, z) as $(\bar{z}, \bar{\tau})$.

field theory with all spatial dimensions infinite. Our correlation functions are now correlation functions with large separations in "time," and their exponential fall-off is determined by the energies of the physical states in this zero-temperature, spatially-periodic field theory. So the following is an exactly equivalent definition:

> *Alternative definition.* Recast the theory as a 3+1 dimensional field theory at zero temperature, where one of the spatial dimensions—call it $\bar{z}$—is periodic with period β. Then, in a Hilbert space interpretation, the Debye mass is the energy of the lightest state that is odd under $\bar{z}$ reflection.

For future reference, we should clarify that we will always use the notation A_0 to denote the component of the gauge field in the periodic direction, regardless of whether we are interpreting Euclidean "time" according to the original definition or the alternative.

In addition to their behavior under $\bar{z}$ reflection, henceforth denoted $\mathcal{R}_{\bar{z}}$, eigenstates of the spatially-periodic 3+1 dimensional theory may also be classified according to their behavior under other space-time symmetries. Specifically, excitations at rest ($p_x = p_y = 0$) can be assigned quantum numbers J^C, $|p_{\bar{z}}|$, and $R_{\bar{z}}$ where J is the angular momentum in the xy-

$J^{C(P)}$	$R_{\bar z}$ odd operators		
0^{-+}	$\mathrm{tr}\,(\mathrm{Im}\Omega) \equiv \mathrm{Im}\,L$	or	$\mathrm{tr}\,(F_{03}\,(F_{12})^2)$
0^{+-}	$\mathrm{tr}\,(\mathrm{Im}\Omega\,F_{12})$	or	$\mathrm{tr}\,(F_{03}\,F_{12})$
0^{--}	$\mathrm{tr}\,(\mathrm{Im}\Omega\,[F_{12},\,(F_{i3})^2])$	or	$\mathrm{tr}\,(F_{03}\,[F_{12},\,(F_{i3})^2])$
0^{++}	$\mathrm{tr}\,(\mathrm{Im}\Omega\,[F_{12},\,[F_{13},F_{23}]])$	or	$\mathrm{tr}\,(F_{03}\,[F_{12},\,[F_{13},F_{23}]])$
1^{+}	$\mathrm{tr}\,(\mathrm{Im}\Omega\,F_{i3})$	or	$\mathrm{tr}\,(F_{03}\,F_{i3})$
1^{-}	$\mathrm{tr}\,(\mathrm{Im}\Omega\,\{F_{12},F_{i3}\})$	or	$\mathrm{tr}\,(F_{03}\,\{F_{12},F_{i3}\})$

TABLE I. Examples of gauge-invariant Euclidean time-reflection (or $\mathcal{R}_{\bar z}$) odd operators which couple to specific $J^{C(P)}$ sectors. Here, $i = 1,2$ is a 2+1 dimensional spatial index and $\Omega = \mathcal{P}\,\exp\int_0^\beta A_0\,dx^0$ denotes the un-traced Wilson line or Polyakov loop. $\mathrm{Im}\Omega$ is shorthand for the anti-Hermitian part, $\mathrm{Im}\Omega \equiv (\Omega - \Omega^\dagger)/2$.

plane, C is charge conjugation, $p_{\bar z}$ is the momentum in the periodic direction $\bar z$, and $R_{\bar z}$ is the sign acquired under $\bar z$ reflection. For $J = 0$, there is also one additional quantum number: the sign P of the state under two-dimensional reflections.[9] The lowest energy states will have $p_{\bar z} = 0$, and our definition of the Debye mass restricts us to $R_{\bar z} = -$; so the states of interest can be summarized by $J^{C(P)}$. It is not clear *a priori* which $J^{C(P)}$ sector will contain the lightest $R_{\bar z}$ odd state.

Gauge invariant operators which couple to specific $R_{\bar z}$ odd symmetry channels may be easily constructed. For example, under $\mathcal{R}_{\bar z}$ reflection $A_0 \to -A_0$ and $L \to L^\dagger$. Hence, the time-reflection odd part of the Wilson line is just the imaginary part. $\mathrm{Im}\,L$ is also odd under charge conjugation, but is even under x- or y-reflections, so the imaginary part of the Wilson line probes the 0^{-+} sector. Table I illustrates some of the possible gauge invariant operators which can be used to probe various symmetry channels. In the language of the effective three-dimensional theory (now to be regarded as 2+1 dimensional), each of these operators creates an A_0 accompanied, because of confinement, by a neutralizing cloud of

[9]The representations of $\mathrm{O}(2) = \mathrm{Z}(2) \times \mathrm{SO}(2)$ are (*i*) two-dimensional representations for each non-zero value of J^2, and (*ii*) two one-dimensional representations, distinguished by their Z(2) charge, for $J^2 = 0$.

glue. The lowest mass state in each of these channels will have a mass of $m_0 + O(g^2T)$. In a direct lattice determination of the Debye mass, one should in principle check these, and perhaps other, channels in order to find the lightest state. (Alternatively, one could consider correlations of operators with less symmetry.)

In pure SU(2) gauge theory,[10] the operation of charge conjugation is in fact an element of the gauge group (namely $i\sigma_2$). Hence, in this theory, any gauge-invariant state must have C even, and so the possible sectors are restricted to the $J^{+(P)}$ channels. (Note that the charge conjugation odd operators shown in table I vanish identically for SU(2), as they must.)

Wilson lines in pure gauge theories

In pure gauge SU(N) theories (that is, gauge theories without matter fields[11]) there is one additional subtlety which occurs with the Wilson line $\mathrm{Im}\, L$, or with more complicated operators containing a Wilson line wrapping around the periodic τ direction. A Euclidean pure gauge theory at non-zero temperature is invariant not only under periodic gauge transformations; it is also invariant under non-periodic gauge transformations that globally multiply the fundamental representation Wilson line L by an element of the center of the gauge group.[12] For SU(N), the center is Z(N), the N-th roots of unity. This Z(N) symmetry is spontaneously broken at high temperature, and there are N different (pure phase) equilibrium states distinguished by the phase of the Wilson line, $\arg \langle L \rangle = 2\pi k/N$, $k = 0, \ldots, N{-}1$. Only one of the equilibrium states (the one in which $\langle L \rangle \approx 1$) is invariant under the naive definition of time reflection. (Each of the other $N{-}1$ equilibrium states

[10]These comments also apply to the $p_{\bar z}$=0 sector of SU(2) theories coupled to fermions (but not complex scalars), since the fermions are irrelevant. In addition, they apply to any group which, like SU(2), has only real (or pseudo-real) representations.

[11]Or more generally, theories whose gauge group has a non-trivial center but all of whose fields transform trivially under the center.

[12]For a review of this symmetry and its role at high temperature, see ref. [9].

is invariant under a re-defined time reflection which combines the original reflection with a non-trivial $Z(N)$ gauge transformation.)

In order for $\mathrm{Im}\, L$ to probe the Debye mass, one must work in the single pure phase equilibrium state which is invariant under (the chosen definition of) time reflection. Otherwise, time reflection will fail to select the charge-screening excitations of interest. However, a gauge theory functional integral which is invariant under the $Z(N)$ center symmetry necessarily averages over all N spontaneously broken phases. Because of this, the $\mathrm{Im}\, L$ correlation length, computed with a $Z(N)$ invariant functional integral will be $O(g^2T)$, and have nothing to do with the real Debye mass. This can be seen directly from the fact that the $\langle LL \rangle$ and $\langle L^\dagger L^\dagger \rangle$ correlations vanish by $Z(N)$ symmetry, and so $\mathrm{Im}\, L$ and $\mathrm{Re}\, L$ will have identical $O(g^2T)$ correlation lengths.

This difficulty does not reflect any inconsistency in our definition of the Debye mass, because $\mathrm{Im}\, L$ is not actually gauge invariant under the full gauge group of pure gauge theories (which includes $Z(N)$ center transformations). Hence, in pure gauge theories, it does not meet the requirements stated in the definition.

Nevertheless, one may avoid this difficulty in pure gauge theories, and obtain an $O(gT)$ correlation length for $\mathrm{Im}\, L$, in either of two ways: change the operator, or change the theory. Fixing the operator is easy: simply replace the fundamental representation trace in the definition of the Wilson line by the trace in some other complex representation F which is invariant under the center of the group. For example, in an $SU(N)$ theory the symmetric tensor product of N fundamental representations is suitable (*i.e.*, the 10 of SU(3)). Then

$$\mathrm{Im}\, L_{\mathrm{F}} \equiv \mathrm{Im}\, \mathrm{tr}\, \mathcal{P} \exp i \int_0^\beta d\tau \; A_0^a(\tau, \underset{\sim}{x}) \, T_{\mathrm{F}}^a \tag{3.4}$$

is invariant under $Z(N)$ transformations and is completely unaffected by the spontaneous breaking of the $Z(N)$ center symmetry. Or, one may use a local operator (of the same symmetry) which does not involve a Wilson line at all, such as $\mathrm{tr}\, (F_{03}\, (F_{12})^2)$.

Alternatively, one may restrict expectation values to include only the single equilibrium state with $\langle L \rangle \approx 1$ by adding an infinitesimal source to the Lagrangian that biases the system

toward the desired Z(N) sector:[13]

$$Z_\epsilon = \int [\mathcal{D}\bar\psi\, \mathcal{D}\psi\, \mathcal{D}A] \exp\left(-\frac{1}{g^2}\int_0^\beta d\tau\, d^3x\, \mathcal{L}_{\rm E} \;+\; \epsilon \int d^3x L \right) , \tag{3.5}$$

and then send ϵ to zero after the infinite volume limit. Therefore, the Im L correlation length ξ may given by

$$\xi^{-1} = -\lim_{|\underset{\sim}{x}|\to 0}\; \lim_{\epsilon\to 0^+}\; \lim_{\mathcal{V}\to\infty}\; |\underset{\sim}{x}|^{-1}\; \ln \langle {\rm Im}\, L(\underset{\sim}{x}) {\rm Im}\, L(0)\rangle_{\epsilon,\mathcal{V}} \,. \tag{3.6}$$

Adding the source term explicitly breaks the Z(N) center symmetry, thereby reducing the full gauge symmetry of the theory so that L is now fully gauge invariant. Of more practical use for numerical simulations, one can simply run simulations in a large enough volume that one finds no jumping between the different Z(N) equilibrium states, as measured by $\langle L\rangle$. One would need to take data generated by a run in a single pure phase where $\bar L \equiv \langle {\rm tr} L\rangle$ is non-zero, and then measure the long-distance fall-off of

$$\langle {\rm Im}\, [L(\underset{\sim}{x})\bar L^\dagger]\; {\rm Im}\, [L(0)\bar L^\dagger]\rangle \tag{3.7}$$

in lieu of $\langle {\rm Im}\, L(\underset{\sim}{x})\; {\rm Im}\, L(0)\rangle$.

Axial theories and chemical potentials

As mentioned earlier, our definition of the Debye mass does not work for theories in which real-time $\mathcal{TC}$ is not a symmetry. Hence, it cannot be applied to gauge theories with axial couplings, or in the presence of a non-zero chemical potential. In the language of our alternative definition, where the z direction is viewed as "time," the problem manifests as follows: the lightest state with a single A_0 is no longer stable against decay into an $\underset{\sim}{A}$ glueball. Nevertheless, there is still a singularity in the complex momentum plane associated with this A_0 "resonance." The situation is depicted in Fig. 7, where we have considered a

[13]The Z(N) center symmetry is explicitly broken in theories with fundamental representation matter fields; hence in such theories no explicit symmetry breaking perturbation need be added.

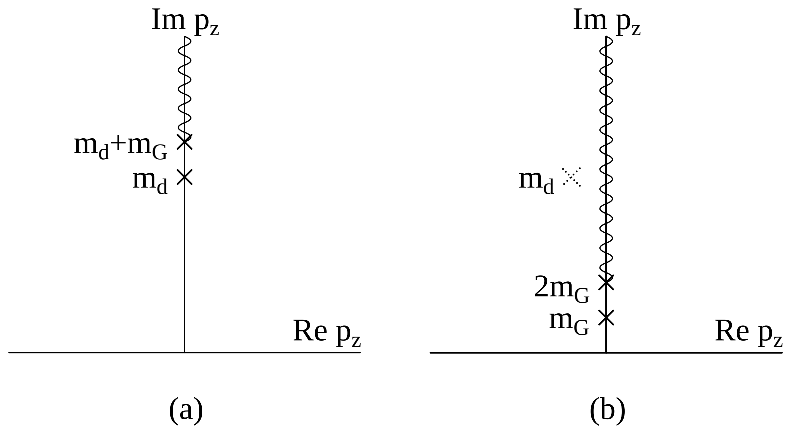

FIG. 7. The singularity structure, in the complex p_z plane, of a correlation of Euclidean time-reflection odd operators in a theory where real-time $\mathcal{TC}$ (a) is, or (b) is not, a good symmetry. $m_{\rm G}$ is $O(g^2T)$ and stands for the lightest glueball mass in the theory when the z direction is regarded as time. For simplicity, we have suppressed all singularities associated with excited glueball or excited Debye states.

generic correlation of Euclidean time-reflection odd operators and sketched some features of its singularity structure in the complex p_z plane. We have assumed $p_\perp = 0$ for simplicity. The case for a $\mathcal{TC}$ conserving theory is shown in Fig. 7(a), where the location of the singularity closest to the real axis is our Debye mass. Introducing a small amount of $\mathcal{TC}$ violation will mix in $\underset{\sim}{A}$ glueball states, changing the analytic structure to that of Fig. 7(b). The Debye singularity from Fig. 7(a) is still present but has moved slightly off the axis onto the second sheet.

One can still imagine, in principle, defining a (complex) Debye mass based on the location of the pole. This is, in fact, what one does every day when talking about the mass of an unstable particle such as the Z-boson or the π^0. There is an important difference, however, which is that the z direction is not really a time direction and can be considered one only by analytic continuation. In the real world, one can reach the Z resonance experimentally by making $-p_0^2 + \underset{\sim}{p}^2$ close to $-M_{\rm Z}^2$. In contrast, one cannot experimentally study the static ($p_0 = 0$) properties of the plasma by taking $\underset{\sim}{p}^2$ close to $-m_{\rm d}^2$. And because the introduction of finite temperature breaks Lorentz invariance, studying $p_0 \neq 0$ instead is not equivalent: the physics of the dynamics of real plasma excitations is not the same as the physics of electric screening. The moral is that defining the Debye mass by the location of the relevant

singularity in Fig. 7(b) would be somewhat abstract.

Another possible method for defining an electric screening length is in terms of carefully chosen moments of particular correlation functions. For the purpose of illustration, assume we had a correlation function that behaved like

$$G(r) = \frac{1}{r}\, e^{-agTr} + \frac{\varepsilon}{r}\, e^{-bg^2Tr}\,. \tag{3.8}$$

The first term is the behavior we would have in a $\mathcal{TC}$ invariant theory; the second term represents the mixing with glueball states due to interactions breaking $\mathcal{TC}$, with ε the amplitude of that mixing. Now consider defining a correlation length by the ratio of moments

$$\xi_G \equiv \frac{\int_0^\infty r^2\, G(r)\, dr}{\int_0^\infty r\, G(r)\, dr}\,. \tag{3.9}$$

This will yield

$$\xi_G = (agT)^{-1} \times \left[1 + O(\varepsilon/g^2)\right]. \tag{3.10}$$

As long as ε is small compared to g^2, this will give a correlation length of order $1/gT$ that, at leading order, matches what we want to call the inverse Debye mass. So one could simply define what one means by the Debye mass to be precisely $1/\xi_G$. The problem with a definition of this form is that it is completely convention dependent; the resulting value depends on exactly which correlation function and which moments are used in the definition. In contrast, lengths characterizing the exponential long-distance decay of correlations, as in our original definition for $\mathcal{TC}$ invariant theories, do not generically depend on the details of the operators used, other than their symmetries.

If the Debye mass is so difficult to define at finite chemical potential, what about the case of non-relativistic QED? Why do physicists generally have few qualms about discussing exponential screening in such plasmas? The reason is a matter of scale and what one means by "long-distance." Ref. [6] gives a calculation of the charge-charge correlation in such plasmas and finds that it does fall algebraically at very large distance. However, as one increases the distance in a variety of physical applications, the correlation first falls

exponentially for many e-foldings before finally tapering off in algebraic behavior, and so the concept of exponential screening is useful in practice.

Now consider the case of relativistic gauge theories and our toy example (3.8) of a correlation $G(r)$. The number of e-foldings over which the first term dominates is only order $\ln(1/\varepsilon)$. Unlike our toy correlation $G(r)$, a real correlation will have additional contributions from *excited* time-reflection odd states, with energies of $m_{\rm d}+O(g^2T)$:

$$G(r) \to \frac{1}{r}\sum O(1)\, e^{-[agT+O(g^2T)]r} + \frac{1}{r}\sum O(\varepsilon)\, e^{-O(g^2T)r} + (\text{other junk})\,. \tag{3.11}$$

In the range where the glueball contributions are small, r is still too small to suppress these excited states unless $g\ln(1/\varepsilon) \gg 1$. Therefore, an approximate definition of the Debye mass, in terms of the intermediate-range fall-off of correlation functions, is not useful beyond leading order unless the amount of $\mathcal{TC}$ violation is extraordinarily small. In particular, it is not useful if the mixing ε is simply some power g^n of the coupling.

IV. THE $O(g^2T)$ CORRECTION TO THE DEBYE MASS

We return now to vectorially-coupled theories at zero chemical potential. If one is interested only in the $O(g^2T)$ correction to the Debye mass, then it is possible to reduce the computation of the Debye mass to a much simpler problem than the extraction of correlation lengths in a four-dimensional theory with a small periodic dimension and dynamical fermions. This simplification will emerge from the successive reduction to equivalent effective theories describing longer distance scales, as discussed in section II. The philosophy is similar to that applied by Braaten [10] to the expansion of the free energy in powers of g. (With more work, it could be extended to handle even higher-order corrections to the Debye mass.) The result, to be derived momentarily, expresses the $O(g^2T)$ part of the Debye mass in terms of the perimeter law coefficient of adjoint-representation Wilson loops in a three-dimensional pure gauge theory. This relation is particularly nice in that it holds regardless of which symmetry channel of the 2+1 dimensional theory has the lowest mass time-reflection (or $\mathcal{R}_{\bar z}$) odd excitations.

First, reduce the problem to an effective three-dimensional theory by integrating out modes with non-zero frequency in the periodic direction. If we relabel the z axis as "time," we want to know the energy of an A_0, together with its cloud of glue, propagating forward in time in 2+1 dimensions. Next, make a further reduction to an effective theory for distances large compared to $1/gT$, so that the bare A_0 can now be considered heavy. The resulting effective theory is simply a 2+1 dimensional pure gauge theory (plus irrelevant corrections suppressed by powers of g). The *non-perturbative* contribution of the cloud of glue surrounding the A_0 is not sensitive to whether the A_0 is merely heavy or is infinitely heavy. The propagation of the bare A_0 can then be replaced by an adjoint-charge Wilson line, exactly analogous to the way in which extremely heavy quarks in zero-temperature QCD can be replaced by fundamental-charge Wilson lines. The non-perturbative piece of the Debye mass is given by the energy of the glue required to screen an infinitely heavy adjoint charge, which can be extracted from a numerical lattice calculation of the perimeter-law behavior of large Wilson loops. Schematically,

$$m_{\rm d} = m_{\rm pert} + \Delta m \,, \tag{4.1}$$

where $m_{\rm pert}$ is a perturbative contribution to the mass and where Δm is extracted from

$$\left\langle {\rm tr}\, \mathcal{P} \exp\left(i \oint_C d\underset{\sim}{x} \cdot \underset{\sim}{A}_{\rm adj} \right) \right\rangle \sim \exp[-\Delta m \, {\rm length}(C)] \,, \qquad \text{for large loops } C, \tag{4.2}$$

in the three-dimensional gauge theory. Note that fermions are completely absent from the calculation of Δm because they decoupled in the three-dimensional limit. Figure 8 illustrates the various stages leading to the relation (4.1), which will be discussed in more detail below.

At each stage in the reduction, it is necessary to match carefully the effective theories onto the original four-dimensional theory, in order to keep track of the perturbative piece $m_{\rm pert}$ of (4.1) correctly. Fortunately, at the order of interest, the matching is fairly simple and straightforward.

Before we dive into the details of matching, notice that this picture of the Debye mass makes the presence of the logarithmic $O(g^2 T \ln(m_0/g^2 T))$ correction found in (1.5), as well

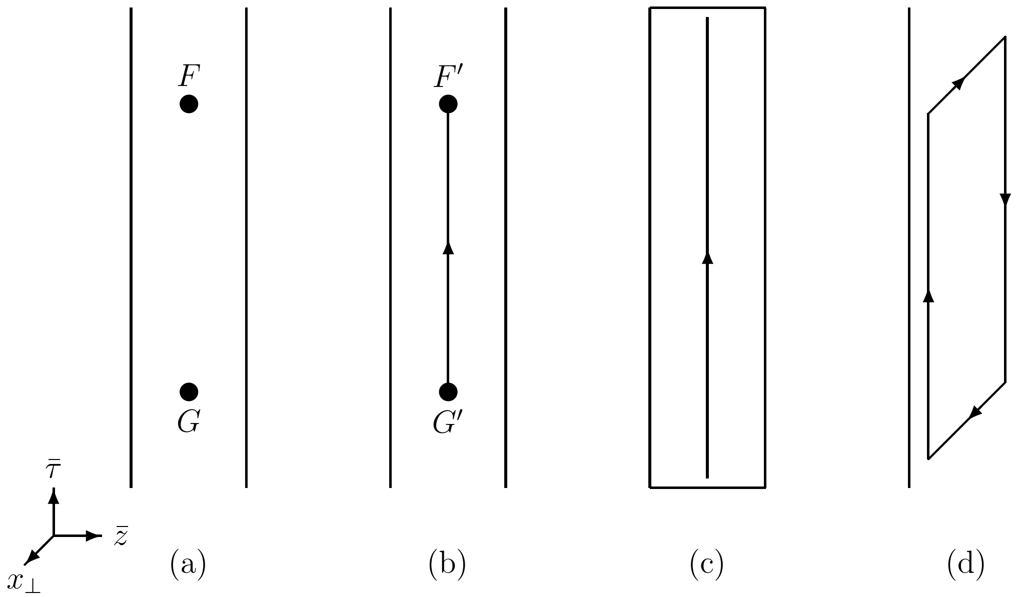

FIG. 8. Various stages in the reduction relating the $O(g^2T)$ correction to the Debye mass to the perimeter law coefficient of three-dimensional adjoint-representation Wilson loops. (a) The correlation of some pair F and G of $\mathcal{R}_{\bar{z}}$ odd operators at large separation. A three-dimensional reduction is performed and only the $p_{\bar{z}} = 0$ modes are relevant to what follows. (b) The A_0 field is integrated out, generating an A_0 propagator connecting modified insertions $F' \sim \partial F/\partial A_0$ and $G' \sim \partial G/\partial A_0$. Since A_0 is heavy, its propagator may be replaced by a straight, adjoint-representation, Wilson line. The long-distance fall-off of the correlation measures the energy of glue needed to screen a heavy adjoint charge. (c) If one makes the $\bar{\tau}$ direction periodic with an arbitrarily large period Θ, then the minimal screening energy of an adjoint charge is determined by the fall-off of an adjoint Wilson line wrapping around the $\bar{\tau}$ direction. (This is not to be confused with the original Polyakov loop L, which wraps around the small $\bar{z}$ direction.) (d) The fall-off of the long adjoint line with increasing length is the same as the perimeter law fall-off of any large, adjoint Wilson loop. (The loop depicted in the figure is meant to be large in $x_\perp$ as well as in $\bar{\tau}$.)

as its coefficient, trivial to understand. Consider the self-energy of a static, infinitely heavy charge in 2+1 dimensions. If we were discussing QED coupled to such a charge, then the electric field surrounding the charge would be $\mathcal{E} = e/2\pi r$, and the energy of that electric field would be

$$E = \tfrac{1}{2} \int d^2x \, \mathcal{E}^2 = \frac{e^2}{4\pi} \int_0^\infty \frac{dr}{r} \,, \tag{4.3}$$

which is logarithmically divergent in both the infrared and the ultraviolet. This same picture holds for a non-Abelian gauge theory at distances small compared to the confinement scale, since then gluon self-interactions are small. The confinement scale, however, provides an infrared cut-off at $r^{-1}{\sim}g^2T$. The mass of the charged particle, which is order m_0 and not actually infinite, provides the ultraviolet cut-off. Finally, since the heavy particle is an adjoint charge, e^2 should be replaced by $C_{\rm A}\, g_3^2$, where $C_{\rm A}$ is the quadratic Casimir number for the adjoint representation, or N for SU(N). Eq. (4.3) then precisely reproduces the logarithm of (1.5).

We now turn to fleshing out the details of the split up (4.1) and matching of the sequence of effective theories. At each stage, an effective theory will describe the same long-distance physics as its shorter-distance predecessor, provided its parameters are carefully matched to the parameters of its predecessor. One can achieve this matching by computing and equating a set of long-distance quantities in both theories. The required matching reflects the different treatment of *short*-distance physics in the two theories; it doesn't depend in detail on the physics at long distances, which in our case is non-perturbative. Hence, if one temporarily modifies the theories by introducing a long-distance cut-off, and one uses the same cut-off for both theories, then the matching of the infrared cut-off theories will *also* provide the correct matching for the theories when the infrared cut-off is removed. The temporary introduction of an infrared cut-off is merely a convenience which allows one to compute the matching perturbatively.

Step 1: Reduction to three dimensions

This reduction has been extensively treated in the literature [10,12,13,14,15]. At the order we are interested in, the matching is very simple. Integrating out the non-static ($p_0 \neq 0$) components of the fields generates a mass term for A_0 through diagrams such as Fig. 3. The effective theory is of the form

$$S_3 = \frac{1}{\bar{g}^2 T} \int d^3x \left[\tfrac{1}{4} F_{ij} F_{ij} + \tfrac{1}{2} (D_{\rm adj} A_0)^2 + \tfrac{1}{2} m_0^2 A_0^2 + (\text{higer-order}) \right] . \tag{4.4}$$

The "higher-order" term denotes marginal and irrelevant operators in the effective theory which are suppressed by explicit powers of g and whose effect can be ignored at the order of interest. The dimensionless coupling $\bar{g}$ is related to the original four-dimensional coupling by

$$\bar{g}^2 = g^2(T) + O(g^4) , \tag{4.5}$$

where evaluating the four-dimensional coupling g at a renormalization scale of order T eliminates large logarithms from the higher-order corrections to this matching condition. The mass m_0 is just, up to $O(g^3 T)$ corrections from two loop diagrams that don't concern us, the mass (1.4) we introduced in the introduction:

$$m_0^2 = \frac{1}{3} \left(C_{\rm A} + \sum_{\rm F} t_{\rm F} \right) g^2 T^2 + O(g^4 T^2) . \tag{4.6}$$

We have been slightly more general than the SU(N) case of (1.4); $C_{\rm A}$ is the quadratic Casimir for the adjoint representation and $t_{\rm F}$ is the normalization of each irreducible fermion representation F:

$$C_{\rm A} \delta^{ab} = f^{acd} f^{bcd} , \qquad t_{\rm F} \, \delta^{ab} = {\rm tr} \left(\mathbf{T}_{\rm F}^a \mathbf{T}_{\rm F}^b \right) . \tag{4.7}$$

We have not yet specified an ultraviolet renormalization scheme for the effective theory (4.4), because it is irrelevant in this step at the order shown above. However we will need a specific scheme in the next step, so let us pick one. We will use dimensional regularization in $d = 3 - 2\epsilon$ dimensions for our effective theory, with the minimal subtraction scheme and a renormalization scale $\mu = T$.

Step 2: Replacing A_0 by a Wilson line

Now we want to integrate out A_0 and move to an effective theory for momenta small compared to m_0. But our goal is to describe the propagation of an A_0 itself, which we might probe by the long-distance behavior of some gauge-invariant correlation in the three-dimensional theory. For example, consider the three-dimensional analog of the first 0^{+-} operator listed in Table I,

$$\left\langle \mathrm{tr}\,(A_0 F_{12})_{(0,\underset{\sim}{x})}\ \mathrm{tr}\,(A_0 F_{12})_{(0,0)} \right\rangle . \tag{4.8}$$

Imagine evaluating this correlation by first doing the path integral over A_0 and only later doing the path integral over $\underset{\sim}{A}$. The integral over A_0 will replace the A_0's appearing in the integrand by a propagator of the A_0 field in the background of $\underset{\sim}{A}$,[14] which is the solution to

$$\left(-D_i^{\mathrm{adj}} D_i^{\mathrm{adj}} + m_0^2\right) \Delta_{m_0,\underset{\sim}{A}}(\underset{\sim}{x}) = \delta^3(\underset{\sim}{x}) . \tag{4.9}$$

Now suppose that $\underset{\sim}{A}$ has only low-momentum components and is smooth on the scale of $1/m_0$. The solution to (4.9) can then be expressed as an expansion in powers of derivatives of $\underset{\sim}{A}$ and gives

$$\Delta_{m_0,\underset{\sim}{A}}(\underset{\sim}{x}) = \Delta_{m_0}(\underset{\sim}{x})\, \mathcal{P} \exp\left(i \int_0^{\underset{\sim}{x}} d\underset{\sim}{x} \cdot \underset{\sim}{A}_{\mathrm{adj}} \right) + O(\partial \underset{\sim}{A}) , \tag{4.10}$$

where the integration path is a straight line from the origin to $\underset{\sim}{x}$, and where Δ_{m_0} is the free scalar propagator which behaves as

$$\Delta_{m_0}(\underset{\sim}{x}) \sim \frac{e^{-m_0|\underset{\sim}{x}|}}{4\pi|\underset{\sim}{x}|} \qquad \text{for } |\underset{\sim}{x}| \to \infty . \tag{4.11}$$

Hence, the net effect of integrating out A_0 will be to replace the pair of A_0's in a correlation such as (4.8) by an adjoint-representation path-ordered exponential, times an overall factor:

[14] This is true in the approximation that we ignore quartic and higher-order interactions among A_0 in the effective Lagrangian (4.4). As mentioned before, such terms are suppressed by explicit powers of the coupling and do not contribute to the Debye mass at the order we are interested. To go to higher orders, one would have to include these terms and treat them as perturbations.

$$\langle A_0(\underset{\sim}{x})A_0(0)\cdots\rangle \to e^{-m_1|\underset{\sim}{x}|}\left\langle \mathcal{P}\exp\Big(i\int_0^{\underset{\sim}{x}} d\underset{\sim}{x}\cdot \underset{\sim}{A}_{\rm adj}\Big)\cdots\right\rangle . \tag{4.12}$$

This corresponds to Fig. 8(b), and naively m_1 is just m_0. The substitution (4.12) will be valid for separations large compared to the inverse Debye mass.

But, as sketched in Fig. 8(c), once one introduces the adjoint representation line, one may dispense with the details of the operator insertions in the original correlation function by considering the $\bar\tau$ direction (the 2+1 dimensional "time") to be periodic with an arbitrarily large period Θ and computing the expectation of a straight adjoint Wilson line wrapping around the $\bar\tau$ direction:

$$e^{-m_1\,\Theta}\left\langle {\rm tr}\,\mathcal{P}\exp\Big(i\int_0^{\Theta} d\bar\tau\cdot \underset{\sim}{A}_{\rm adj}\Big)\right\rangle . \tag{4.13}$$

In the Hilbert space interpretation, this corresponds to a trace over all states containing a single adjoint-representation external charge. (And includes a sum over all $J^{C(P)}$ sectors.) The exponential fall-off of this expression with the period Θ will be determined by the energy of the lightest such state — which is precisely our definition of the Debye mass.

Finally, as indicated in Fig. 8(d), the same coefficient for the fall-off of the correlation with contour length may instead be obtained by considering a large, topologically trivial, adjoint loop:

$$e^{-m_1\,|C|}\left\langle {\rm tr}\,\mathcal{P}\exp\Big(i\oint_C d\underset{\sim}{x}\cdot \underset{\sim}{A}_{\rm adj}\Big)\right\rangle . \tag{4.14}$$

The exponential fall-off of this expression with contour length $|C|$ will yield the Debye screening length through $O(g^2T)$. The perimeter-law decay of the Wilson loop gives the energy of the glue surrounding the heavy adjoint charge, while m_1 above is the mass of the bare charge.

There is just one complication: the substitution (4.10) is only a good approximation in the presence of gauge fields $\underset{\sim}{A}$ with small momentum. Our two effective theories—the one with A_0 and the one where we've replaced it by a Wilson loop—differ in how they treat large momentum effects. As usual, this means we need to carefully adjust parameters in order

to make the two theories describe the same long-distance physics. In particular, the correct choice of m_1 in (4.14) is not necessarily m_0; it must be determined by matching. This is done by perturbatively computing the long-distance fall-off of the A_0 correlation functions in both theories, which we will only need to do at one-loop order. Dimensional regularization will be used as our infrared cut-off.

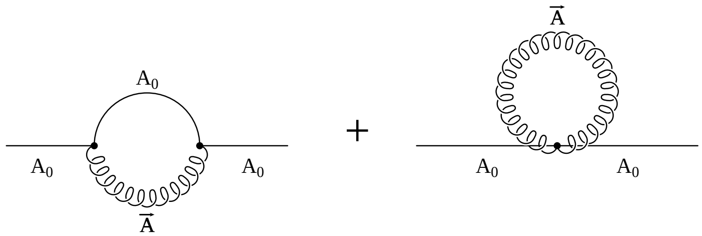

FIG. 9. The one-loop self-energy for A_0 in the three-dimensional theory.

In the original three-dimensional effective theory, the fall-off of the A_0 propagator is determined by the position of the pole. The shift in the pole position due to the one-loop self energy can be computed from the diagrams of Fig. 9 and gives a contribution to the Debye mass of

$$\delta m_0^2 = C_{\rm A}\, \bar{g}^2 \mu^{1+2\epsilon} \int \frac{d^{3-2\epsilon} q}{(2\pi)^{3-2\epsilon}} \left\{ \frac{1}{q^2+m_0^2} + \frac{(1-2\epsilon)}{q^2} + \frac{2(m_0^2-p^2)}{q^2[(p+q)^2+m_0^2]} \right. \\ \left. \left. + (\xi-1)(p^2+m_0^2) \frac{q^2+2p\cdot q}{q^4[(p+q)^2+m_0^2]} \right\} \right|_{p^2=-m_0^2}, \tag{4.15}$$

which yields

$$\delta m_0 = \frac{1}{8\pi}\, C_{\rm A}\, \bar{g}^2 \mu \left[-\frac{1}{\epsilon} + \ln\left(\frac{m_0^2}{\pi\mu^2}\right) + \gamma_{E} - 1 \right]. \tag{4.16}$$

We have worked in covariant gauge with gauge parameter ξ.

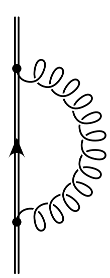

FIG. 10. The one-loop self-energy for a Wilson line.

Correspondingly, to obtain the fall-off of an adjoint-charge Wilson line we need to com-

pute the one-loop self-energy correction for a static source, as shown in Fig. 10. Locally, we can treat a very large Wilson loop as straight. Alternatively, we can confine the system to a very large but finite volume and let a straight Wilson line wrap periodically around the space, as in Fig. 8(c). In any case, taking the line to be in the z direction, Fig. 10 gives

$$\delta m_1 = C_{\rm A}\, \bar{g}^2 \mu \int_0^\infty dz\, \Delta_{zz}(z, \underset{\sim}{0}) = 0\,. \tag{4.17}$$

where $\Delta_{ij}(z, \underset{\sim}{x}_\perp)$ is the $\underset{\sim}{A}$ propagator and the integral vanishes in dimensional regularization.

Matching requires that $m_0 + \delta m_0 = m_1 + \delta m_1$. Putting (4.16) and (4.17) together, and continuing to choose our renormalization scale in each effective theory to be $\mu = T$, gives

$$m_1 = m_0 + \frac{1}{8\pi}\, C_{\rm A}\, \bar{g}^2 T \left[-\frac{1}{\epsilon} + \ln\left(\frac{m_0^2}{\pi T^2}\right) + \gamma_{\scriptscriptstyle E} - 1 \right]. \tag{4.18}$$

The coupling $\bar{g}$ here still represents the coupling $g(T)$ in the original four-dimensional theory. The fact that the matching of couplings between effective theories becomes non-trivial beyond leading order won't enter into our results at the order of interest.

Step 3: From the continuum to the lattice

To measure the perimeter-law behavior of large adjoint Wilson loops numerically, one will put the system on a lattice.[15] The ultraviolet will be regulated by the lattice spacing instead of by dimensional regularization, and so we need to modify our matching condition (4.18) appropriately. We will do this by again matching the one-loop self-energy, now between an adjoint line in the continuum and one on the lattice. However, dimensional regularization is no longer a good choice for our temporary, common, infrared cut-off in the two theories. Instead, we shall consider two, opposite, parallel Wilson lines running in the z direction and separated by a large distance R in x. This provides an infrared cut-off because the lines

[15]If the lattice theory is defined with link variables U in some representation (typically the fundamental representation), then the adjoint Wilson loop is given by the path-ordered product of $U_{\rm adj}$ over the links of the loop, where $U_{\rm adj}^{ab}$ is $t_{\rm F}^{-1}\,{\rm tr}(T^a U T^b U^\dagger)$ for each link.

neutralize each other when viewed from large $\underset{\sim}{x}_\perp = (x, y)$. The one-loop contribution to the energy of these lines is shown in Fig. 11.

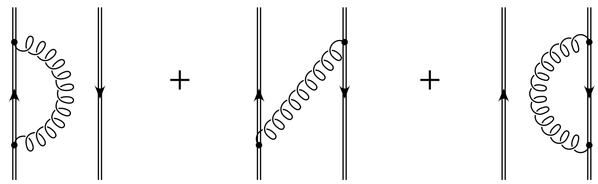

FIG. 11. The one-loop self-energy for two opposing Wilson lines, separated by a distance R.

In the continuum, the result is

$$
\begin{aligned}
\delta' m_1 &= C_\mathrm{A}\, g^2 \mu^{1+2\epsilon} \int_0^\infty dz\ [\Delta_{zz}(z, \underset{\sim}{0}) - \Delta_{zz}(z, \underset{\sim}{R})] \\
&= \frac{1}{2} C_\mathrm{A}\, g^2 \mu^{1+2\epsilon} \int \frac{d^{2-2\epsilon} p_\perp}{(2\pi)^{2-2\epsilon}} \frac{1}{p_\perp^2} \left(1 - e^{i\underset{\sim}{p}_\perp \cdot \underset{\sim}{R}}\right) \\
&= \frac{1}{8\pi}\, C_\mathrm{A}\, g^2 \mu \left[\frac{1}{\epsilon} + \ln(\pi \mu^2 R^2) + \gamma_E \right] + O(\epsilon)\,. \qquad (4.19)
\end{aligned}
$$

In lattice perturbation theory, the result is similar but we get lattice propagators instead of continuum ones:

$$
\delta m_\mathrm{lat} = \frac{1}{2} C_\mathrm{A}\, g^2 a^{-1} \int \frac{d^2 p_\perp}{(2\pi)^2}\, \Delta_\mathrm{lat}(p_\perp) \left(1 - e^{i p_x R/a}\right)\,, \qquad (4.20)
$$

where a is the lattice spacing; $p_\perp$ is in lattice units and restricted to the Brillouin zone $|p_x|, |p_y| < \pi$; and

$$
\Delta_\mathrm{lat}(p_\perp) = \frac{1}{4} \left[\sin^2 \frac{p_x}{2} + \sin^2 \frac{p_y}{2} \right]^{-1}\,. \qquad (4.21)
$$

The $R \gg a$ limit of the integral is extracted in the appendix and gives

$$
\delta m_\mathrm{lat} \to \frac{1}{8\pi}\, C_\mathrm{A}\, g^2 a^{-1} \left[\ln\!\left(8R^2/a^2\right) + 2\gamma_E \right]\,. \qquad (4.22)
$$

To match the theories, we must pick $a^{-1} = \mu = T$ to make the coupling constant definition match up, and then require $m_\mathrm{lat} + \delta m_\mathrm{lat} = m_1 + \delta' m_1$, or

$$
m_\mathrm{lat} = m_0 + \frac{1}{8\pi}\, C_\mathrm{A}\, g^2 T \left[\ln\left(\frac{m_0^2}{8T^2}\right) - 1 \right]\,. \qquad (4.23)
$$

The remaining contribution to the Debye mass is now the quantity extracted from the perimeter-law fall-off of large adjoint Wilson loops on the lattice. Since we want specifically to extract the coefficient of the $O(g^2T)$ contribution to the Debye mass, we only need the leading-order result for the perimeter-law exponent in the limit that g is small. Formally, this is

$$\Delta m \sim g^2 T \lim_{g\to 0} \lim_{|C|\to\infty} \left[-\frac{1}{g^2|C|} \ln \left\langle \mathrm{tr}\,\mathcal{P} \exp\Bigl(i \oint_C d\underset{\sim}{x} \cdot \underset{\sim}{A}_{\mathrm{adj}} \Bigr) \right\rangle \right] , \tag{4.24}$$

and $|C|$ is the perimeter of the loop in lattice units. The limit, however, diverges logarithmically as $g{\to}0$ because of the physics behind the logarithm in (1.5) and (4.3). To cure the problem, we simply need to extract this logarithm explicitly and combine it with the perturbative contribution m_{lat}:

$$m_{\mathrm{d}} = m_0 + g^2 T \left\{ \alpha + \frac{C_{\mathrm{A}}}{8\pi} \left[\ln\left(\frac{m_0^2}{8 g^4 T^2} \right) - 1 \right] \right\} + O(g^3 T) \, , \tag{4.25}$$

where

$$\alpha = \lim_{g\to 0} \lim_{|C|\to\infty} \left[-\frac{1}{g^2|C|} \ln \left\langle \mathrm{tr}\,\mathcal{P} \exp\Bigl(i \oint_C d\underset{\sim}{x} \cdot \underset{\sim}{A}_{\mathrm{adj}} \Bigr) \right\rangle + \frac{C_{\mathrm{A}}}{8\pi} \ln\left(g^4 \right) \right] . \tag{4.26}$$

This is our final result for the $O(g^2T)$ contribution to the Debye mass expressed in terms of the perimeter law coefficient for large adjoint-representation Wilson loops in three-dimensional pure lattice gauge theory. All that is needed to obtain a numerical result is for someone to compute the value of α on the lattice for gauge theories of interest [namely SU(3) and SU(2)].

We thank Alexei Abrikosov, Lowell Brown, and David Kaplan for useful conversations. This work was supported by the U.S. Department of Energy, grant DE-FG06-91ER40614.

APPENDIX

To do the integral in (4.20), first do the p_y integral, which is straightforward and gives

$$I(r) \equiv \int \frac{d^2 p_\perp}{(2\pi)^2} \, \Delta_{\mathrm{lat}}(p_\perp) \left(1 - e^{i p_x r} \right) = \frac{1}{2\pi} \int_0^{\pi/2} dq \, \frac{1 - \cos(2rq)}{\sin q \sqrt{1 + \sin^2 q}} \, , \tag{A1}$$

where $q \equiv p_x/2$. Now split the integral into two pieces:

$$I(r) = \frac{1}{2\pi}\int_0^{\pi/2} dq \left[\frac{1}{\sin q\sqrt{1+\sin^2 q}} - \frac{1}{q}\right][1-\cos(2rq)] + \frac{1}{2\pi}\int_0^{\pi/2}\frac{dq}{q}[1-\cos(2rq)]. \quad \text{(A2)}$$

When $r \to \infty$, the first term can be replaced by

$$\begin{aligned}\frac{1}{2\pi}\int_0^{\pi/2} dq \left[\frac{1}{\sin q\sqrt{1+\sin^2 q}} - \frac{1}{q}\right] &= \frac{1}{2\pi}\left[\frac{1}{4}\ln\left(\frac{1-\sqrt{1-\sin^4 q}}{1+\sqrt{1-\sin^4 q}}\right) - \ln q\right]\Bigg|_{q=0}^{\pi/2} \\ &= \frac{1}{2\pi}\left[-\ln\pi + \tfrac{3}{2}\ln 2\right]. \end{aligned} \quad \text{(A3)}$$

The second integral in (A2) is straightforward, and the final result is

$$I(r) \sim \frac{1}{4\pi}\left[\ln(8r^2) + 2\gamma_E\right] \qquad \text{as } r \to \infty. \quad \text{(A4)}$$

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